INTEGRAL EQUATIONS RELATED TO THE REPRESENTATION OF FUNCTIONS BY POTENTIALS

BY

JOSEPH HOPE HORNBACK

A.B., Central College (Missouri), 1932
A.M., Harvard University, 1933

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS
IN THE GRADUATE COLLEGE OF THE
UNIVERSITY OF ILLINOIS, 1952

URBANA, ILLINOIS
1952

INTEGRAL EQUATIONS RELATED TO THE REPRESENTATION OF FUNCTIONS BY POTENTIALS

BY

JOSEPH HOPE HORNBACK

A.B., Central College (Missouri), 1932
A.M., Harvard University, 1933

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS
IN THE GRADUATE COLLEGE OF THE
UNIVERSITY OF ILLINOIS, 1952

URBANA, ILLINOIS
1952

R. UNIV.
BIBLIOTHEEK
LEIDEN

INTEGRAL EQUATIONS RELATED TO THE REPRESENTATION OF FUNCTIONS BY POTENTIALS

A work of W. J. Trjitzinsky[1] contains several theorems concerning representations of the form

$$(I)\quad F(x,y) = (-1/2\pi) \iint_W [f(u,v)\ \log(1/r)]\ du\ dv + h(x,y).$$

In the above equation, W denotes a bounded, open domain in an ordinary euclidean space of two dimensions; the points (x,y) and (u,v) are in W; r is the distance between (x,y) and (u,v,); and h(x,y) is a function which is harmonic in the domain W. The function f(u,v) is shown to be a sort of Laplacian of F(x,y). An equation of the form (I) gives rise to a problem in integral equations as follows: given a function F(x,y) of a suitable class, find a function f(u,v), where f(u,v) is to be of a certain class, such that substitution in equation (I) defines a function h(x,y) which is harmonic in W. Similar problems for complex valued functions defined in a bounded region of the complex plane, with the addend an analytic function, have also been considered by Trjitzinsky.[2]

The present paper generalizes the above problem by introducing a kernel into the integrand. The problem is to start with a kernel k(x,y;u,v) and a function F(x,y) of suitable classes and solve the integral equation

$$(II)\quad \iint_W [k(x,y;u,v)\ f(u,v)\ \log(1/r)]\ du\ dv - F(x,y) = h(x,y)$$

for a function f(u,v) of a certain class so that h(x,y) is harmonic in W. Conditions used are such that f(u,v) is a solution of a regular Fredholm integral equation of the second kind. By methods of proof analogous to those used by Trjitzinsky[3] for functions of complex variables, conditions are placed upon F(x,y) and k(x,y;u,v) which make equation (II)

equivalent to a regular Fredholm integral equation in the sense that every solution of either is a solution of both. Since the theory of the Fredholm equation is known, there is thereby obtained information about the solutions of equation (II).

The fundamental theorem of the paper establishes some quite general, although somewhat implicit, conditions under which a problem in the representation of a function by a generalized potential is equivalent to a regular integral equation. Some conditions are concerned mainly with commutativity of operations: the flux of an integral equals the integral of the flux, and the "ordinary" derivate of an integral equals the integral of the derivate. Other conditions require certain functions to be of class L_2 and others to be of a class T, defined as those which satisfy the hypotheses of Trjitzinsky's Theorem 5.6[1], which is basic to many of the considerations of the present work. This fundamental theorem is essentially an outline of the method used subsequently in considering various classes of functions. This method is applied first to the class of functions F(x,y) which satisfy, in W, a uniform Lipschitz condition and have continuous partial derivatives of the first order.

The major results of the present paper are found in the considerations of the functions of classes $C[_{\rho}(n)]$ and G of Trjitzinsky[1]. Here the derivates of the flux of F(x,y) are allowed to be essentially unbounded. The investigation closes with the consideration of a class of functions having second derivatives.

1. Trjitzinsky, W. J., "Mixed Laplacians and Potential Representations," **Annali di Matematica pura ed applicata,** 31 (1950), 143-230.

2. Trjitzinsky, W. J., "Problems of Representation and Uniqueness for Functions of a Complex Variable," **Acta Mathematica,** 78 (1946), 97-192.

3. Trjitzinsky, W. J., "Integral Equations in Problems of Representation of Functions of a Complex Variable," **Duke Mathematical Journal,** 12 (1945), 419-440.

VITA

Joseph Hope Hornback was born on April 20, 1910, at Nevada, Missouri. He received his elementary and secondary education in the public schools of that town. His undergraduate college work was done at Central College, Fayette, Missouri, where the degree of Bachelor of Arts was conferred in 1932. He then entered Harvard University for graduate work and received the degree of Master of Arts the following year. He did additional work at the University of Chicago before going to Calumet City, Illinois, where he taught mathematics at Thornton Fractional Township High School from 1934 to 1937. He was a teacher of mathematics and critic teacher at the Laboratory School of the University of Chicago from 1937 until his enlistment in the United States Navy in 1942. He taught mathematics at Great Lakes Naval Training Station for a little more than a year and was then commissioned. He received training at Fort Schuyler, New York, Princeton University, Massachusetts Institute of Technology, and Corpus Christi, Texas, before being assigned to duty at the Bureau of Aeronautics of the Navy Department, Washington, D. C. He also served at the Naval Research Laboratory before his release to inactive duty in 1946 as a Lieutenant. Following this release, he did further graduate work at the University of Chicago. He entered the Graduate College of the University of Illinois in 1949, and gave instruction in mathematics as a part time assistant from that time until February 1952. He is a member of Sigma Xi.